On Close - to - Convex Function and Univalent Function

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ABSTRACT
Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and \( g(z) \) be regular in the unit disk \( E = \{ z : |z| < 1 \} \). In this study, a condition under which \( \text{Re} \frac{f'(z)}{g(z)} > 0 \) was established and the result applied to discuss the univalency of the function \( f(z) \). AMS (MOS) Subject classification codes 30C45, 30C50.

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INTRODUCTION
Let \( A \) denote the class of functions of the form \( f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \) which are regular in the unit disk \( |z| < 1 \). We denote the subclass of \( A \) consisting of regular and univalent functions \( f(z) \) in the unit disk and satisfying \( \text{Re} \frac{f'(z)}{g(z)} > 0 \) by \( S_0 \).

Definitions
1. A function \( f(z) \) is said to be convex in the unit disk \( |z| < 1 \) if and only if \( \text{Re} \left[ 1 + \frac{f'(z)}{f(z)} \right] > 0 \).
2. A function \( f(z) \) is said to be starlike in the unit disk \( |z| < 1 \) if and only if \( \text{Re} \left[ z \frac{f'(z)}{f(z)} \right] > 0 \).
3. A function \( f(z) \) regular in the unit disk \( |z| < 1 \) is said to be close-to-convex function if there is a convex function \( g(z) \) such that \( \text{Re} \left[ \frac{f'(z)}{g(z)} \right] > 0 \).
It is clearly known that if $\Re \left[ \frac{f'(z)}{g'(z)} \right] > 0$ for $|z| < 1$, and then $f(z)$ is close-to-convex. So it is natural to be interested in the condition under which $\Re \left[ \frac{f'(z)}{g'(z)} \right] > 0$ for $|z| < 1$ clearly, every starlike function is close-to-convex function. It is known in MacGregor (1963) that if $\Re \left[ \frac{f'(z)}{g'(z)} \right] > \frac{1}{2}$ for $|z| < 1$ then $f(z)$ is starlike for $|z| < 2^{-\frac{1}{4}}$. Many authors have worked in this direction, for example, in Marx (1932) and Strohhacker (1993). It was shown that if $f(z)$ is convex in $|z| < 1$ then $\Re \left[ \frac{f(z)}{g(z)} \right] > \frac{1}{2}$ (Miller and Mocanu, 1978). Also, in Strohhacker (1993) it was proved that if $f(z)$ satisfies $\Re(z) > \beta$ for $0 \leq \beta < 1$ and $z \in E$ then $\Re \left[ \frac{f(z)}{g(z)} \right] > \frac{1+2\beta}{3}$.

This was improved in Yamagushi (1966). This result was also improved in Opoola and Fadipe-Joseph (2009), Babalola and Opoola (2007) where the condition under which $\Re \left[ \frac{f(z)}{g(z)} \right] > \frac{1}{2}$ was established.

Preliminary and statement of results

Let $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$, $|z| < 1$ and let $h(w, x) = f \left( g^{-1}(w) \right)$ be a function in the unit disk. Also let

$$g(w) = \frac{w'(z + x)(1 + |x|^2)}{f'(\frac{x + z}{1 + z}) - f(x)}$$

It is known that $f(z)$ is univalent for $|z| < 1$ if $g(z)$ is univalent for $|z| < 1$.

Let $z = g^{-1}(w)$ then $g(z) = w$.

Also let $g(w) = \frac{1}{z} + h(w, x) = f^{-1}(h(w, x))$.

It is known that in Nehari (1949) that $f(z)$ is univalent for $|z| < 1$ if and if $|h(z, x)| < 1$ for $|x| < 1$ $|z| < 1$. The following lemmas shall be used to prove the main result.

Lemma [5]

If $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \in S_0$ then the partial sums.
\[ S_n(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n \quad (n = 2, 3, \ldots) \]

are univalent in \(|z| < \frac{1}{4}\)

Lemma [8]

If \( f(z) \in S_0 \) then \( f(z) \) is univalent in \(|z| < 2^{\frac{1}{2}-1}\)

Lemma [8] - Wolff-Noshiro’s lemma (Noshiro, 1934)

If \( f(z) \) is analytic in \([z] < R \) and \( \text{Re} f'(z) > 0 \) \((|z| < R)\) then \( f(z) \) is univalent in \(|z| < R\)

The main result in this paper is the following theorem.

Theorem

If \( f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \) is regular for \(|z| < 1\) and

\(|h(w, x)| < 1\) for \(|z| < 1, |x| < 1\) then \( \text{Re}\left[\frac{f(z)}{g(z)}\right] > \frac{1}{2}\)

**Proof of the main result and applications**

In this section, the proof of the main result in this paper is given. Applications of the main results were also given.

Proof of the Theorem 2.4.

Let \( f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \) be regular for \(|z| < 1\). Consider the function

\[ g(w) = \frac{w'(z+x)[1+|x|^2]}{f\left(\frac{xz}{1+x^2}\right) - f(x)} = \frac{1}{z} + h(w, x) \]

Since \( f(0) = 0 \) and \( f'(0) = 1 \) we have that

\[ h(w, 0) = \frac{w'(z)}{f'(z)} - \frac{1}{z} \]

By the condition of the theorem and since \(|z| < 1\) we have that

\[ \left|\frac{w'(z)}{f'(z)} - \frac{1}{z}\right| < 1 \]

which implies that

\[ \text{Re}\left[\frac{f(z)}{g(z)}\right] > \frac{1}{2} \]
Since \( w'(z) = g'(z) \).

As application we prove the following theorems.

Theorem
If \( f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \) is regular for \( |z| < 1 \)
and \( |h(w, x)| < 1 \) for \( |z| < 1; |x| < 1 \),
then the partial sums
\[
S_t(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_t z^t \quad (t = 2, 3, \ldots)
\]
are univalent in \( |z| < \frac{1}{4} \)

Proof:
By theorem (2.4) we have that
\[
\text{Re} \left( \frac{f(z)}{g(z)} \right) > \frac{1}{2}
\]
Hence, by applying a theorem of Yamagushi (1966), it follows that the partial sums
\[
S_t(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_t z^t \quad (t = 2, 3, \ldots)
\]
are univalent in the unit disk \( |z| < \frac{1}{4} \)
since every starlike function is a close-to-convex function.

3.2 Theorem
If \( f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \) is regular for \( |z| < 1 \) and \( |h(w, x)| < 1 \) for \( |z| < 1; |x| < 1 \),
then
\[
\text{Re} \left( f'(r e^{i\theta}) \right) \geq \frac{1-2r-r^2}{(1+r)^2}
\]
for \( 0 \leq r < 2 \frac{1}{2} - 1 \)

Proof:
Let \( f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \) is regular for \( |z| < 1 \) and \( |h(w, x)| < 1 \) for \( |z| < 1; |x| < 1 \),
then
\[
\text{Re} \left( \frac{f(z)}{g(z)} \right) > \frac{1}{2}
\]
by theorem (2.4). Hence by a theorem of Yamagushi (1966), we have that
\[
\text{Re} \left( f'(r e^{i\theta}) \right) \geq \frac{1-2r-r^2}{(1+r)^2}
\]
for \( 0 \leq r < 2 \frac{1}{2} - 1 \) since every starlike function is a close-to-convex function. The bound is sharp.

Theorem
If $f(z) = z + z + a_2 z^2 + a_3 z^3 + \ldots$ is regular for $|z| < 1$ and $|h(w,x)| < 1$ for $|z| < 1$; $|x| < 1$, then $f(x)$ is univalent for $|z| < 2 \frac{1}{2} - 1$

**Proof:**

Let $f(z) = z + z + a_2 z^2 + a_3 z^3 + \ldots$ is regular for $|z| < 1$ and $|h(w,x)| < 1$ for $|z| < 1$, $|x| < 1$, then

$$\text{Re} \left( \frac{f'(z)}{z f'(z)} \right) > \frac{1}{2}$$

by theorem (2.4). Also by theorem (3.2)

$$\text{Re} \left( f'(re^{i\theta}) \right) \geq \frac{1-2r-r^2}{(1+r)^2}$$

for $0 \leq r < 2 \frac{1}{2} - 1$. Which implies that

$$\text{Re} \left( f'(re^{i\theta}) \right) > 0$$

for $0 \leq r < 2 \frac{1}{2} - 1$ since $g'(z) > 0$. The result follows from the well known Wolff-Noshiro’s lemma (Noshiro, 1934).

**REFERENCES**


