A-Stable Order Eight Second Derivative Linear Multistep Methods for Solutions of Stiff Systems of Ordinary Differential Equations (ODEs)

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ABSTRACT

In this study, accurate and efficient numerical methods with good stability properties shall be developed. The formulation of the block second derivative Blended Linear Multistep methods for step numbers k=7 is considered. The main methods are derived by blending of two methods by continuous collocation approach. These methods are of uniform order eight. With this approach, we hope to improve the stability regions of the Adams Moultons Methods with step number k=7 and thereby making them suitable for the solution of stiff ordinary differential equations. The new methods proposed in this paper turn out to be A-stable. Numerical examples obtained demonstrate the accuracy and efficiency of the new Blended Block Linear Multistep Methods.

Keywords *A Stability, Blended Linear Multistep Methods, Adams Methods, Backward Differentiation Method (BDF), Stiff ODEs,*
MSC Numerical Mathematics

INTRODUCTION

Most real life problems when modelled mathematically result in ordinary differential equations. Some of the equations do not have analytic solutions as such the need for good numerical methods to approximate their solutions. In this paper we are concern with the numerical solution of the stiff initial value problem (1) using the second derivative linear multistep.

$$y'(x) = f(x, y(x)), y(x_0) = y_0 \qquad \text{.. (1)}$$

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on the finite interval

where $y:[x_0, x_N] \rightarrow \Re^m$ and $f:[x_{0,} x_N] \times \Re^m \rightarrow \Re^m$

is continuous and differentiable.

The second derivative k-step method takes the following form

$$\sum_{i=0}^{k} a_j y_{n+j} = h\sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=0}^{k} \gamma_j g_{n+j} \qquad \text{.. (2)}$$

Where a_j, b_j and g_j are parameters to be determined and $g_{n+j} = f_{n+j}^{\cdot}$.

Several methods have been developed to overcome this barrier theorem. Researchers like Gear (1965), Butchers (1966), Lambert (1973), the second derivative methods of Enright (1974), Genin (1974), Gamal and Iman (1998), Sahi, Jator and Khan (2012), Mehdizadeh, Nasehi and Hojjati (2012), Ehigie and Okunuga (2014) and the third derivative method of Ezzeddine and Hojjati (2012), Chollom and Omagwu (2016), either relax the condition to obtain A−stable methods or incorporate off-step points to improve the stability of the methods.

In this work, we consider the second derivative hybrid explicit generalized Adams methods for step numbers k=7. With this approach, we hope to improve the stability regions of the Adams Moulons Methods and thereby making them suitable for the solution of (1).

Formulation of the Method

Let m = 8 in (2) produces the general form of the Blended Block Linear Multistep method for k=7 as:

$$\tilde{\tilde{y}}(x_{n+7}) = a_6 x_{n+6} + h\begin{bmatrix} b_0(x)f_n + b_1(x)f_{n+1} + b_2(x)f_{n+2} + b_3(x)f_{n+3} + \\ b_4(x)f_{n+4} + b_5(x)f_{n+5} b_6(x)f_{n+6} + b_7(x)f_{n+7} \end{bmatrix} +$$

$$h^2\lambda_7(x)y''_{n+7} \qquad \text{....................................(3)}$$

$$D = \begin{bmatrix} 1 & (x_n+6\mathrm{h}) & (x_n+6\mathrm{h})^2 & \cdots & (x_n+6\mathrm{h})^8 \\ 0 & 1 & 2\mathrm{x_n} & \dots & 11\mathrm{x_n}^7 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2(x_n+7\mathrm{h}) & \dots & 8(x_n+7\mathrm{h})^7 \\ 0 & 0 & 2 & \cdots & 56(x_n+7\mathrm{h})^6 \end{bmatrix}$$

……………………………….(4)

$$\begin{aligned}\bar{y}(\tau+x_n) = y_{n+6} &+ \left(\tau - \frac{1159\tau^2}{840h} + \frac{1531\tau^3}{1512h^2} - \frac{3839\tau^4}{8640h^3} + \frac{2647\tau^5}{21600h^4} - \frac{139\tau^6}{6480h^5} + \frac{\tau^7}{432h^6} - \frac{17\tau^8}{120960h^7} - \frac{17\tau^9}{1272160h^8} - \frac{199h}{700}\right) f_n \\
&+ \left(\frac{21\tau^2}{5h} - \frac{739\tau^3}{150h^2} - \frac{3221\tau^4}{1200h^3} - \frac{7547\tau^5}{9000h^4} + \frac{299\tau^6}{1440h^5} - \frac{457\tau^7}{25200h^6} + \frac{11\tau^8}{9600h^7} - \frac{\tau^9}{32400h^8} - \frac{1413h}{875}\right) f_{n+1} \\
&+ \left(\frac{-63\tau^2}{8h} + \frac{949\tau^3}{80h^2} - \frac{601\tau^4}{80h^3} + \frac{12449\tau^5}{4800h^4} - \frac{19\tau^6}{36h^5} + \frac{71\tau^7}{1120h^6} - \frac{\tau^8}{240h^7} + \frac{\tau^9}{8640h^8} + \frac{27h}{350}\right) f_{n+2} \\
&+ \left(\frac{35\tau^2}{3h} - \frac{10194\tau^3}{54h^2} + \frac{5617\tau^4}{432h^3} - \frac{5213\tau^5}{1080h^4} + \frac{2701\tau^6}{2592h^5} - \frac{397\tau^7}{3024h^6} - \frac{31\tau^8}{3456h^7} + \frac{\tau^9}{3888h^8} - \frac{89h}{35}\right) f_{n+3} \\
&+ \left(\frac{-105\tau^2}{8h} + \frac{527\tau^3}{24h^2} - \frac{3037\tau^4}{192h^3} + \frac{8881\tau^5}{1440h^4} - \frac{67\tau^6}{48h^5} + \frac{185\tau^7}{1008h^6} - \frac{5\tau^8}{384h^7} + \frac{\tau^9}{5292h^8} - \frac{99h}{140}\right) f_{n+4}\end{aligned}$$

$$+\left(\frac{63\tau^2}{5h}-\frac{43\tau^3}{2h^2}+\frac{1273\tau^4}{80h^3}-\frac{3847\tau^5}{600h^4}+\frac{2167\tau^6}{1440h^5}-\frac{23\tau^7}{112h^6}-\frac{29\tau^8}{1920h^7}-\frac{\tau^9}{2160h^8}-\frac{459h}{175}\right)f_{n+5}$$

$$+\left(\frac{679\tau^2}{120h}+\frac{104933\tau^3}{10800h^2}-\frac{26113\tau^4}{3600h^3}+\frac{212701\tau^5}{72000h^4}-\frac{379\tau^6}{540h^5}+\frac{4889\tau^7}{50400h^6}+\frac{13\tau^8}{1800h^7}-\frac{29\tau^9}{129600h^8}-\frac{401h}{1750}\right)f_{n+6}$$

$$+\left(\frac{3\tau^2}{7h}+\frac{157\tau^3}{210h^2}-\frac{137\tau^4}{240h^3}+\frac{431\tau^5}{1800h^4}-\frac{17\tau^6}{288h^5}+\frac{43\tau^7}{5040h^6}-\frac{3\tau^8}{4480h^7}-\frac{\tau^9}{45360h^8}+\frac{9h}{175}\right)f_{n+7}$$

$$+\left(\frac{7\tau^2}{2h}-\frac{121\tau^3}{20h}+\frac{3283\tau^4}{720h^2}-\frac{6769\tau^5}{3600h^3}+\frac{49\tau^6}{108h^4}-\frac{23\tau^7}{360h^5}+\frac{7\tau^8}{1440h^7}-\frac{\tau^9}{6480h^8}-\frac{9h}{25}\right)g_{n+7}$$

$$g_{n+7}=y''_{n+7}$$

……………………………….(5)

The above continuous formulation (5 is then evaluated at the following points to give the following seven discrete schemes which are used simultaneously for the solution of (1) constitute the block method.

$$y_n = y_{n+6} - \frac{199}{700}hf_n - \frac{1413}{875}hf_{n+1} + \frac{27}{350}hf_{n+2} - \frac{89}{35}hf_{n+3} + \frac{99}{140}hf_{n+4}$$

$$-\frac{459}{175}hf_{n+5} + \frac{401}{1750}hf_{n+6} + \frac{9}{175}hf_{n+7} - \frac{9}{25}h^2y''_{n+7}$$

$$y_{n+1} = y_{n+6} + \frac{1625}{217728}hf_n - \frac{25685}{72576}hf_{n+1} - \frac{250}{189}hf_{n+2} - \frac{141875}{217728}hf_{n+3}$$

$$y_{n+2} = y_{n+6} - \frac{52}{42525}hf_n + \frac{1328}{70875}hf_{n+1} - \frac{1892}{4725}hf_{n+2} - \frac{10288}{8505}hf_{n+3}$$

$$- \frac{2356}{2835}hf_{n+4} - \frac{5392}{4725}hf_{n+5} - \frac{30928}{70875}hf_{n+6} + \frac{16}{14175}hf_{n+7} + \frac{104}{2025}h^2y''_{n+7}$$

$$y_{n+3} = y_{n+6} + \frac{11}{22400}hf_n - \frac{657}{112000}hf_{n+1} + \frac{27}{700}hf_{n+2} - \frac{2021}{4480}hf_{n+3}$$

$$-\frac{5031}{4480}hf_{n+4} - \frac{20871}{22400}hf_{n+5} - \frac{7321}{14000}hf_{n+6} - \frac{99}{22400}hf_{n+7} + \frac{81}{800}h^2y''_{n+7}$$

$$y_{n+4} = y_{n+6} - \frac{29}{170100}hf_n + \frac{139}{70875}hf_{n+1} - \frac{107}{9450}hf_{n+2} + \frac{421}{8505}hf_{n+3}$$

$$-\frac{5237}{11340}hf_{n+4} - \frac{5321}{4725}hf_{n+5} - \frac{64003}{141750}hf_{n+6} - \frac{1}{2025}hf_{n+7} + \frac{127}{2025}h^2y''_{n+7}$$

$$y_{n+5} = y_{n+6} + \frac{1201}{5443200}hf_n - \frac{20609}{9072000}hf_{n+1} + \frac{52}{4725}hf_{n+2} - \frac{37631}{1088640}hf_{n+3} + \frac{32233}{362880}hf_{n+4} - \frac{302429}{604800}hf_{n+5} - \frac{633277}{1134000}hf_{n+6} - \frac{8563}{1814400}hf_{n+7} + \frac{7297}{64800}h^2y''_{n+7}$$

$$y_{n+7} = y_{n+6} - \frac{6031}{5443200}hf_n + \frac{99359}{9072000}hf_{n+1} - \frac{943}{18900}hf_{n+2} - \frac{153761}{1088640}hf_{n+3} - \frac{105943}{362880}hf_{n+4} + \frac{341699}{604800}hf_{n+5} + \frac{449527}{1134000}hf_{n+6} + \frac{416173}{1814400}hf_{n+7} + \frac{33953}{64800}h^2y''_{n+7}$$

Stability Analysis of the methods (Ehigie and Okunugha 2014):

The seven step method has order $(8, 8, 8, 8, 8, 8, 8)^T$ and error constant of

Zero-stability of the Block Methods

Following the work of Ehigie and Okunuga (2014), we observed that the seven step block method is zero stable as the roots of the equation

are less than or equal to 1. Since the block method is consistent and zero-stable, the method is convergent (Henrici 1962).

Region of Absolute Stability

Solving the characteristic equation $\left(\det\left(r\left(A - Cz - \frac{D}{z^2}\right) - B\right)\right)$ for r, we obtain the stability function (R(z)):

$$R(z) = \left[\frac{181440(3348z^6 + 73656z^5 - 8015677z^4 - 107373258z^3 + 585043713z^2 + 8709120}{118296521280z^8 - 398268079200z^7 - 1203674098815z^6 + 17140481870582z^5 + 203807488019508z^4 - 83077823942880z^3 - 286454982461760z^2 + 152180571916800z + 1580182732800} \right]$$

Numerical Experiments

Problem 5.1 Stiff Linear System

$$y_1' = 998y_1 + 1998y_2$$
$$y_2' = -999y_1 - 1999y_2$$
$$y_1(0) = 1, y_2(0) = 1$$

Problem 5.2 Van der pol's Equations

The *Van der Pol's Equation* is an important kind of second-order non-linear auto-oscillatory equation. It is a non-conservative oscillator with non-linear damping.

$$0 \le x \le 40, \quad h = 0.1$$
$$y_1' = y_2$$
$$y_2' = -y_1 - \mu y_2(1 \cdot$$
$$\mu = 40, y_1(0) = 2,$$

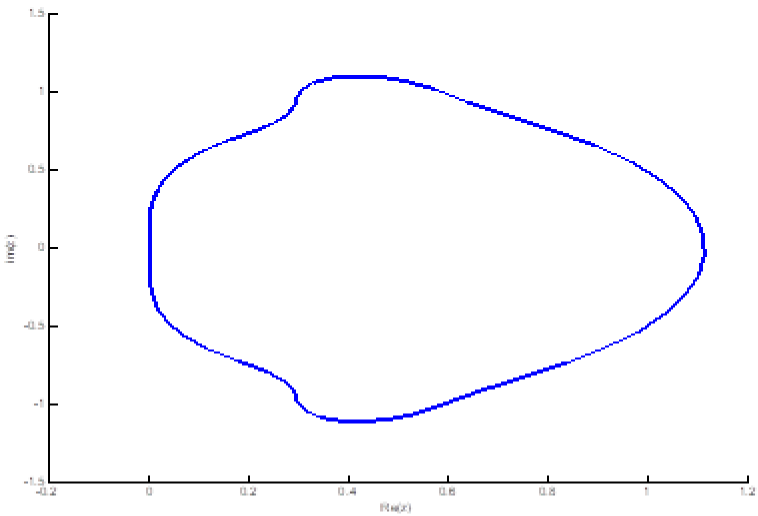
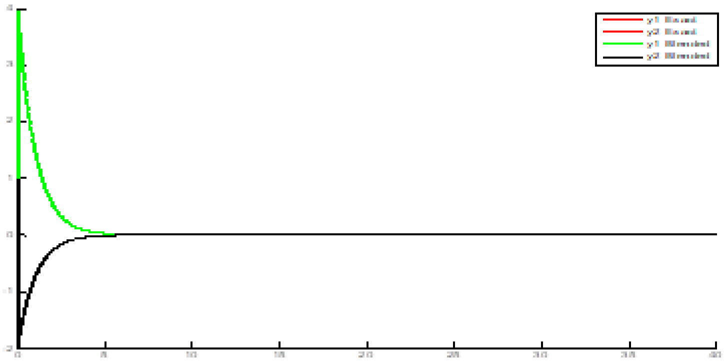
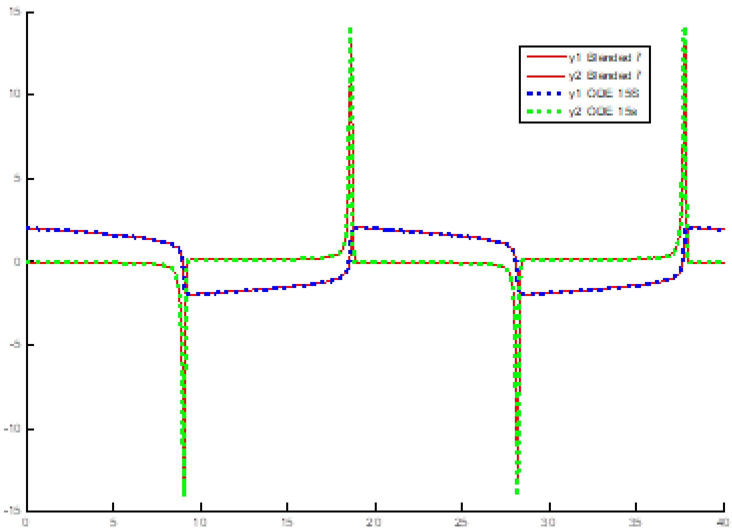


Figure 1: Region of Absolute Stability of the BBLMM for K=7



Solution Curve For Problem 5.1



Solution Curve For Problem 5.2

CONCLUSION

New blended block second derivative linear multistep methods have been constructed through the multistep collocation approach for the solution of stiff systems. The analysis of the stability properties shows that the methods are all A-stable and convergent. Numerical experiments reveal from the solution curves that the efficiency and accuracy of the newly constructed Blended Block Linear multistep methods compete favourably with the variable step size ODE 15s for solving IVPS of Ordinary Differential Equations.

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